DECAY

Simulating decay chains using spreadsheets

The vast majority of radioactive parent isotopes do not decay immediately to stable daughter isotopes. Radioisotopes usually undergo a series of intermediate decays, creating a decay chain. For example, the neutron-rich fission fragments produced in nuclear reactors usually undergo a large number of β^- decays in order to reach stability; a fact that is exploited by a new method for discovering clandestine nuclear reactors [4].

The mathematics of decay chains

Harry Bateman first analysed radioactive decay chains in 1910 [2], but Bateman's paper is no longer easily available and so for the sake of completeness an abridged derviation is given below.

In this example we will consider the decay of neutron-rich lead-211 to lead-207 via a three-stage process. (Note that the decay of bismuth-211 to polonium-211 only occurs in 0.276% of cases; decay to thallium-207, and then to lead-207, is far more common.)

$$^{211}\mathrm{Pb} \xrightarrow{\beta} {}^{211}\mathrm{Bi} \xrightarrow{\beta} {}^{211}\mathrm{Po} \xrightarrow{\alpha} {}^{207}\mathrm{Pb}\,.$$

Lead-211 has a half-life of 36.1 minutes and therefore a decay constant of $\lambda_1 = 3.20 \times 10^{-4}\,\mathrm{s}^{-1}$; bismuth-211 has a half-life of 128 s and therefore a decay constant of $\lambda_2 = 5.42 \times 10^{-3}\,\mathrm{s}^{-1}$; polonium-211 has a half-life of 516 ms and therefore a decay constant of $\lambda_3 = 1.34\,\mathrm{s}^{-1}$; and lead-207 is stable.

$R_{\mathrm{Bi}}(t)$ is the rate at which bismuth-211 is formed at time t, where t is some time between the start of decay and now (time T):

$$R_{\mathrm{Bi}}(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\left(N_0 \mathrm{e}^{-\lambda_1 t}\right) = \lambda_1 N_0 \mathrm{e}^{-\lambda_1 t}.$$

This of course is the negative of the rate of decay of the lead-211, where N_0 is the initial number of $^{211}\mathrm{Pb}$ nuclei.

$n_{\mathrm{Bi}}(t)$ is the amount of bismuth-211 formed in an instant δt at time t:

$$n_{\mathrm{Bi}}(t) = R(t)\,\delta t$$
$$n_{\mathrm{Bi}}(t) = \lambda_1 N_0 \mathrm{e}^{-\lambda_1 t}\,\delta t\,.$$

This decays by time T to

$$R(t)\,\delta t\,\mathrm{e}^{-\lambda_2 (T-t)} = \lambda_1 N_0 \mathrm{e}^{-\lambda_1 t} \mathrm{e}^{-\lambda_2 (T-t)}\,\delta t\,.$$

Integrating for all t between $t = 0$ and $t = T$ gives the population of $^{211}\mathrm{Bi}$ at time T:

$$N_{\mathrm{Bi}}(T) = \int_0^T \lambda_1 N_0 \mathrm{e}^{-\lambda_1 t} \mathrm{e}^{-\lambda_2 (T-t)}\,\mathrm{d}t$$
$$= \lambda_1 N_0 \mathrm{e}^{-\lambda_2 t} \int_0^T \mathrm{e}^{-\lambda_1 t} \mathrm{e}^{\lambda_2 t}\,\mathrm{d}t$$
$$= \lambda_1 N_0 \mathrm{e}^{-\lambda_2 T} \int_0^T \mathrm{e}^{(\lambda_2 - \lambda_1) t}\,\mathrm{d}t$$
$$= \frac{\lambda_1}{\lambda_2 - \lambda_1} N_0 \left(\mathrm{e}^{-\lambda_1 T} - \mathrm{e}^{\lambda_2 T}\right).$$

Extending this process to the decay of Bi-211 to Po-211 gives

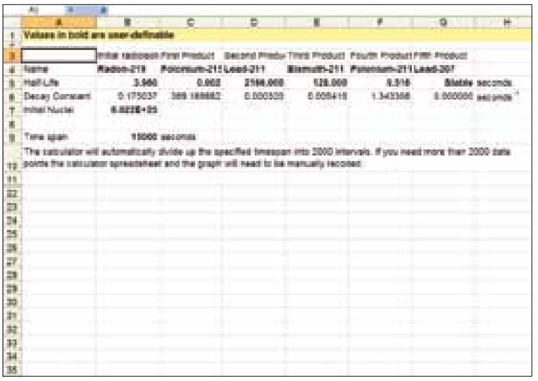

Figure 1. *The spreadsheet as seen in Microsoft Excel. An OpenDocument format (.ODS) version is also available.*

$$N_{\mathrm{Po}}(T) = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} N_0 \mathrm{e}^{-\lambda_3 T} \left(\frac{1}{\lambda_3 - \lambda_1}\left(\mathrm{e}^{(\lambda_3 - \lambda_1) T} - 1\right) - \frac{1}{\lambda_3 - \lambda_2}\left(\mathrm{e}^{(\lambda_3 - \lambda_2) T} - 1\right)\right)$$
$$= \lambda_1 \lambda_2 N_0 \left(\frac{\mathrm{e}^{-\lambda_1 T}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} + \frac{\mathrm{e}^{-\lambda_2 T}}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} + \frac{\mathrm{e}^{-\lambda_3 T}}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}\right).$$

A particularly able (and patient) student, especially one also studying mathematics, may be able to derive this result from first principles. Pointing out that the rate of formation of polonium-211 is equal to the rate of decay of bismuth-211 less the rate of decay of lead-211 may help.

This suggests that the population of lead-207 is

$$N_{\mathrm{Pb}}(T) = \lambda_1 \lambda_2 \lambda_3 N_0 \left(\frac{\mathrm{e}^{-\lambda_1 T}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)} + \frac{\mathrm{e}^{-\lambda_2 T}}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)} + \frac{\mathrm{e}^{-\lambda_3 T}}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}\right).$$

This mathematical process can be extended as far as necessary, to accomodate as many steps as exist in the decay chain [3]. The population of the nth isotope in a decay chain at time t, assuming that there is no independent production of the ith daughter isotope, is given by

$$N_n(t) = \sum_{i=1}^{i=n}\left(\left(\prod_{j=i}^{j=n-1} \lambda_{(j,j+1)}\right) \times \sum_{j=i}^{j=n}\left(\frac{N_{i0}\mathrm{e}^{-\lambda_j t}}{\prod_{p=i,p\neq j}^{p=n}(\lambda_p - \lambda_j)}\right)\right)$$

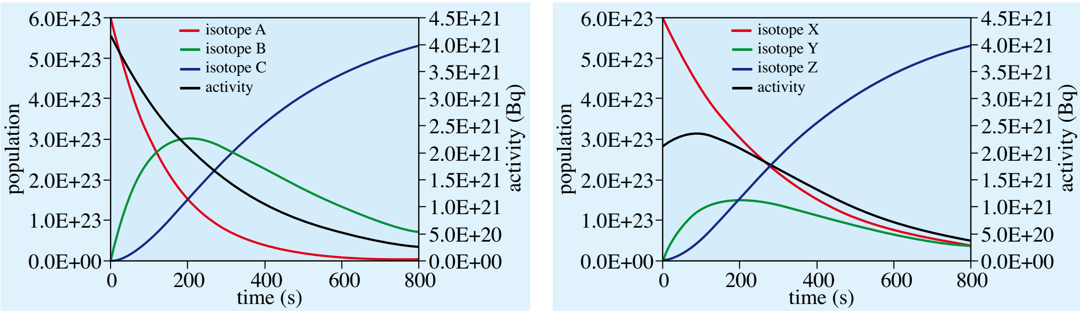

Figure 2. *A comparison of the decay chains of 1 mole each of isotope-A (left) and isotope-X (right). Note that on both graphs activity is plotted on the right-hand axis.*

The reader may notice that singularities occur in the unlikely case that two decay constants are equal. While there is a solution to this problem [4] it is not possible to implement in spreadsheet software and is beyond the scope of this paper.

Implementation

A Microsoft Excel and OpenOffice.org spreadsheet implementation of the calculations above, created by the authors, is available at http://wordpress.MrReid.org/decay-chains/. This spreadsheet is initially configured for the decay chain of lead-211, but it can be reconfigured by altering values shown in bold. The spreadsheet makes use of a number of hidden cells; these are used to calculate the intermediate coefficients outlined in the derivation above.

In these examples we consider the decay chains of two imaginary elements: isotope-A and isotope-X.

$$A \rightarrow B \rightarrow C$$

The half-life of isotope-A is 100 s, of isotope-B is 200 s and isotope-C is stable. That is, the daughter has a half-life that is twice that of the parent.

$$X \rightarrow Y \rightarrow Z$$

The half-life of isotope-X is 200 s, of isotope-Y is 100 s and isotope-Z is stable. That is, the daughter has a half-life that is half that of the parent.

It is interesting to compare how the populations of each isotope vary over the same period of time. Over a long enough period someone ingesting equal amounts of isotope-A and isotope-X would receive the same dose, assuming equal decay energies. Isotope-A would be more likely to cause acute radiation syndrome (radiation sickness) because its

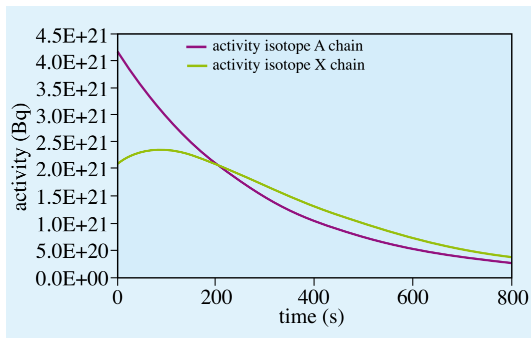

Figure 3. *A comparison of the total activity over time of 1 mole each of isotope-A and isotope-X.*

initial activity is higher. The activity of isotope-X initially increases in activity before decreasing.

Acknowledgement

The authors would like to thank an anonymous referee for comments on an earlier draft of this paper.

References

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